ON THE CUBIC EQUATION WITH FIVE UNKNOWNS

$$3(x^3 - y^3) = z^3 - w^3 + 12t^2 + 4$$

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ABSTRACT

The non-homogeneous Cubic Equation with five unknowns represented by $3(x^3 - y^3) = z^3 - w^3 + 12t^2 + 4$ *is analyzed for its patterns of non–zero integral solutions. Six different patterns of non-zero distinct integer solutions are obtained. A few interesting properties between the solutions and special number patterns namely Polygonal numbers, Centered Polygonal numbers, Pyramidal numbers, Stella Octangular numbers, Star numbers and Pentatope number are exhibited.*

KEY WORDS

Cubic Equation with Five Unknowns, Integral solutions

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INTRODUCTION

Integral solutions for the homogeneous (or) non homogeneous Diophantine cubic equation is an interesting concept as it can be seen from [1, 2, 3]. In [4-9] a few special cases of cubic Diophantine equation with 4 unknowns are studied. In [10, 11], cubic equations with 5 unknowns are studied for their integral solutions. In this communication we present the integral solutions of an interesting cubic equation with 5 unknowns $3(x^3-y^3)=z^3-w^3+12t^2+4$. A few remarkable relations between the solutions are presented.

NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- $ct_{m,n}$ - Centered polygonal number of rank n with size m.
- gn_a - Gnomonic number of rank a
- so_n - Stella octangular number of rank n
- s_n - Star number of rank n
- pr_n - Pronic number of rank n
- pt_n - Pentatope number of rank n

METHOD OF ANALYSIS

The Cubic Diophantine equation with five unknowns to be solved for getting non-zero integral solutions is

$$3(x^3-y^3)=z^3-w^3+12t^2+4 \tag{1}$$

On substituting the linear transformations

$$x=c+1,\ y=c-1,\ z=a+1,\ w=a-1 \tag{2}$$

in (1), it leads to

$$3c^2=a^2+2t^2 \tag{3}$$

In what follows, we present six ways of solving (3) and in view of (2), six different solutions patterns to (1) are obtained.

PATTERN-1

Take $c=p^2+2q^2$ (4)

Write 3 as $3=(1+i\sqrt{2})(1-i\sqrt{2})$ (5)

Substituting (4) & (5) in (3) and using the method of factorization, define

$$(a+i\sqrt{2}t)=(1+i\sqrt{2})(p+i\sqrt{2}q)^2 \tag{6}$$

Equating the real and imaginary parts of (6), we get

$$a(p,q)=p^2-2q^2-4pq \tag{7}$$

$$t(p,q) = p^2 - 2q^2 + 2pq \quad (8)$$

Substitute (4) & (7) in (2) ,the integral solutions of (1) are given by

$$x(p,q) = p^2 + 2q^2 + 1$$

$$y(p,q) = p^2 + 2q^2 - 1$$

$$z(p,q) = p^2 - 2q^2 - 4pq + 1$$

$$w(p,q) = p^2 - 2q^2 - 4pq - 1$$

$$t(p,q) = p^2 - 2q^2 + 2pq$$

PROPERTIES:

- $x(1,q) - z(1,q) = 8t_{3,q}$
- $t(p,p+1) - w(p,p+1) = ct_{12,p}$
- $t(p,p-1) - w(p,p-1) = s_p$
- $x(p,1) + t(p,1) - 1 = 2pr_p$
- $2t(2p,1) - x(2p,1) - y(2p,1) = 8gn_{p^2}$

Each of the following represents a nasty number

a) $3\{x(p,p) - z(p,p)\}$ b) $3\{y(p,p) - w(p,p)\}$ c) $t(p,p) - w(p,p) - 1$

d) $t(p,p) - z(p,p) + 1$ e) $6\{2t(3p,p) - x(3p,p) - y(3p,p)\}$

PATTERN-2:

In equation (5), Writing 3 as $3 = \frac{(5+i\sqrt{2})(5-i\sqrt{2})}{9}$

The corresponding values of a and t satisfying (3) are

$$a(p,q) = \frac{1}{3}(5p^2 - 10q^2 - 4pq)$$
$$t(p,q) = \frac{1}{3}(p^2 - 2q^2 + 10pq) \quad (9)$$

Since our aim is to find the integral solutions, it is seen that a and t are integers for suitable choices of p and q.

CHOICE -1:

Let $p = 3P, q = 3Q$

The corresponding integral solutions of (1) are obtained as below

$$x(P,Q) = 9P^2 + 18Q^2 + 1$$

$$y(P,q) = 9P^2 + 18Q^2 - 1$$

$$z(P,Q) = 15P^2 - 30Q^2 - 12PQ + 1$$

$$w(P,Q) = 15P^2 - 30Q^2 - 12PQ - 1$$

$$t(P,Q) = 3P^2 - 6Q^2 + 30PQ$$

CHOICE -2:

Let $p = 3P + 1, q = 3Q + 1$

For this choice, the non-zero integral solutions of (1) are found to be

$$x(P,Q) = 9P^2 + 18Q^2 + 6P + 12Q + 4$$

$$y(P,q) = 9P^2 + 18Q^2 + 6P + 12Q + 2$$

$$z(P,Q) = 15P^2 - 30Q^2 + 6P - 24Q - 12PQ - 2$$

$$w(P,Q) = 15P^2 - 30Q^2 + 6P - 24Q - 12PQ - 4$$

$$t(P,Q) = 3P^2 - 6Q^2 + 12P + 6Q + 30PQ + 3$$

PROPERTIES:

- $3t(p,p) - x(p,p) = 3gn_{3p^2}$
- $15t(p(p+1), 2p+1) - 3w(p(p+1), 2p+1) - 3 = 324p_p^4$

- $3\{5t(p,(p+1)(p+2)) - z(p,(p+1)(p+2)) + 1\} = 324p_p^3$
- $3\{10t(p,2p^2+1) - z(p,2p^2+1) - w(p,2p^2+1)\} = 108OH_p$
- $2t(2p,1) - x(2p,1) - y(2p,1) = 8gn_{p^2}$

Each of the following represents a nasty number

a) $5x(p,p) - 3z(p,p) - 2$ b) $5x(p,p) - 3w(p,p) - 8$

c) $15t(p,p) - 3w(p,p) - 3$ d) $3t(p,p) - x(p,p) + 1$

e) $z(p,-p) + w(p,-p)$

PATTERN-3:

In equation (5), Writing 3 as $3 = \frac{(1+i11\sqrt{2})(1-i11\sqrt{2})}{81}$

Repeating the above process, we've

$$a(p,q) = \frac{1}{9}(p^2 - 2q^2 - 44pq)$$

$$t(p,q) = \frac{1}{9}(11p^2 - 22q^2 - 2pq)$$

CHOICE-1:

Let $p = 3P, q = 3Q$

The corresponding integral solutions of (1) are obtained as below

$$x(P,Q) = 9P^2 - 18Q^2 + 1$$

$$y(P,q) = 9P^2 - 18Q^2 - 1$$

$$z(P,Q) = P^2 - 2Q^2 - 44PQ + 1$$

$$w(P,Q) = P^2 - 2Q^2 - 44PQ - 1$$

$$t(P,Q) = 11P^2 - 22Q^2 - 2PQ$$

PROPERTIES:

- $x(p,p+1) - 9z(p,p+1) + 8 = 44pr_p$

- $y(p,p(p+1)) - 9z(p,p(p+1)) + 10 = 88p_p^5$
- $9\{t(p,2p^2-1) - 11z(p,2p^2-1) + 1\} = 402SO_p - 2$
- $y(p,q) - 9w(p,q) \equiv 8 \pmod{44}$
- $x(p,q) + y(p,q) - 9z(p,q) - 9w(p,q) \equiv 0 \pmod{88}$

Each of the following represents a nasty number

a) $-3\{x(p,p) + y(p,p)\}$

b) $30\{z(-2p,p) + w(-2p,p)\}$

c) $6\{x(-p,p) + y(-p,p) + z(-p,p) - 1\}$

PATTERN – 4

Rewriting (3) as $3c^2 - a^2 = 2t^2$ (10)

Assume $t = 3p^2 - q^2$ (11)

Write 2 as $2 = (\sqrt{3}+1)(\sqrt{3}-1)$ (12)

Substitute (11) & (12) in (10) and using the method of factorization, define

$$(\sqrt{3}c + a) = (\sqrt{3}+1)(\sqrt{3}p + q)^2$$

Equating the rational and irrational parts , we get

$$a(p,q) = 3p^2 + q^2 + 6pq$$

$$c(p,q) = 3p^2 + q^2 + 2pq$$

The corresponding integral solutions of (1) are given by

$$x(p,q) = 3p^2 + q^2 + 2pq + 1$$

$$y(p,q) = 3p^2 + q^2 + 2pq - 1$$

$$z(p,q) = 3p^2 + q^2 + 6pq + 1$$

$$w(p,q) = 3p^2 + q^2 + 6pq - 1$$

$$t(p,q) = 3p^2 - q^2$$

PROPERTIES:

- $z(p(p+1),(p+2)(p+3)) - x(p(p+1),(p+2)(p+3)) = 96pt_p$
- $z(p,(p+1)) + w(p,(p+1)) - x(p,(p+1)) - y(p,(p+1)) = 16t_{3,p}$
- $w(p(p+1),p+2) - y(p(p+1),p+2) = 24p_p^3$
- $t(p,q) + w(p,q) = \text{Nasty number} + gn_{3pq}$
- $t(p,1) + w(p,1) + 1 = 6pr_p$

Each of the following represents a nasty number

a) $6\{w(p,p) - y(p,p)\}$ b) $2\{t(p,p) + w(p,p) + 1\}$

c) $21\{t(p,p) + x(p,p) + y(p,p)\}$ d) $6\{y(p,p) + z(p,p)\}$

PATTERN – 5

Write 2 as $2 = (3\sqrt{3}+5)(3\sqrt{3}-5)$

Repeating the above process the non-zero distinct integral solutions of (1) are

$$x(p,q) = 9p^2 + 3q^2 + 10pq + 1$$
$$y(p,q) = 9p^2 + 3q^2 + 10pq - 1$$
$$z(p,q) = 15p^2 + 5q^2 + 18pq + 1$$
$$w(p,q) = 15p^2 + 5q^2 + 18pq - 1$$
$$t(p,q) = 3p^2 - q^2$$

PROPERTIES:

- $x(p+1,p) - 3t(p+1,p) - 1 = \text{Nasty number} + 10pr_p$
- $3z(p,q) - 5x(p,q) = 2gn_{pq}$
- $y(p,q) + z(p,q) \equiv 0(\text{mod}\,4)$
- $3w(p,2p^2-1) - 5y(p,2p^2-1) - 2 = 4SO_p$

Each of the following represents a nasty number

a) $6\{x(p,p)-3t(p,p)-1\}$

b) $5x(p,p)+y(p,p)+3t(p,p)-3z(p,p)-1$

PATTERN – 6

Consider (3) as $a^2 = 3c^2 - 2t^2$ (13)

Substitute the linear transformations,

$$c = p + 2q, t = p + 3q \tag{14}$$

in (13), we have $a^2 = p^2 - 6q^2$

which is satisfied by

$$p = 6r^2 + s^2$$
$$q = 2rs$$
$$a = 6r^2 - s^2$$

Hence , the corresponding non-zero integral solutions of (1) are seen to be

$$x(r,s) = 6r^2 + s^2 + 4rs + 1$$

$$y(r,s) = 6r^2 + s^2 + 4rs - 1$$

$$z(r,s) = 6r^2 - s^2 + 1$$

$$w(r,s) = 6r^2 - s^2 - 1$$

$$t(r,s) = 6r^2 + s^2 + 6rs$$

PROPERTIES:

- $t(3r, r-1) - y(3r, r-1) = S_r$
- $t(r,s) - x(r,s) = gn_{rs}$
- $2t(r, r(r+1)) - x(r, r(r+1)) - y(r, r(r+1)) = 8P_r^5$
- $x(r,s) + w(r,s) \equiv 0(\mathbf{mod}\, 4)$

Each of the following represents a nasty number

a) $2\{x(r,r)+y(r,r)-z(r,r)-w(r,r)\}$

b) $6\{2t(r,r)-x(r,r)-y(r,r)\}$

REMARKABLE OBSERVATIONS:

Employing the solutions (x,y,z,w,t) of (1), a few observations among the special polygonal and pyramidal numbers are exhibited below

1. $\left[\frac{3P_z^3}{t_{3,z}}\right]^3-\left[\frac{P_w^5}{t_{3,w}}\right]^3+4\equiv 0(\mathbf{mod}\,3)$

2. $\left[\frac{6P_z^4}{t_{6,z+1}}\right]^3-\left[\frac{6P_{w-1}^4}{t_{3,2w-2}}\right]^3\equiv 2(\mathbf{mod}\,3)$

3. $\left[\frac{P_{z-1}^4}{t_{3,z-1}}-\frac{P_{z-1}^3}{t_{3,z}}\right]^3-\left[\frac{6P_{w-1}^4}{t_{3,2w-2}}\right]\equiv 2(\mathbf{mod}\,3)$

4. $6\left\{\left[\frac{6P_x^4}{t_{6,x+1}}\right]^3-\left[\frac{4Pt_{y-3}}{p_{y-3}^3}\right]^3\right\}-2\left[\frac{3P_z^3}{t_{3,z}}\right]^3+2\left[\frac{4P_w^5}{ct_{4,w}-1}\right]-8$ is a nasty number

5. $3\left[\frac{3p_{x-1}^4-p_{x-1}^3}{t_{3,x-2}}\right]^3-3\left[\frac{36p_{y-2}^3}{s_{y-1}-1}\right]^3+\left[\frac{t_{3,2w-1}}{gn_w}\right]^3-12\left[\frac{12p_{t-2}^5}{s_{t-1}-1}\right]^2-4$ is a cubical integer

CONCLUSION:

To conclude, one may search for other patterns of solutions and their corresponding properties.

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